# Equivalent effective Lagrangians for Scherk-Schwarz compactifications

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**Abstract**

We discuss the general form of the mass terms that can appear in the effective field theories of coordinate-dependent compactifications *à la* Scherk-Schwarz. As an illustrative example, we consider an interacting five-dimensional theory compactified on the orbifold $S^1/Z_2$, with a fermion subject to twisted periodicity conditions. We show how the same physics can be described by equivalent effective Lagrangians for periodic fields, related by field redefinitions and differing only in the form of the five-dimensional mass terms. In a suitable limit, these mass terms can be localized at the orbifold fixed points. We also show how to reconstruct the twist parameter from any given mass terms of the allowed form. Finally, after mentioning some possible generalizations of our results, we re-discuss the example of brane-induced supersymmetry breaking in five-dimensional Poincaré supergravity, and comment on its relation with gaugino condensation in M-theory.

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Traditionally, the effective field theories of coordinate-dependent compactifications *à la* Scherk-Schwarz [1], where the periodicity conditions on the fields are twisted by a symmetry of the action, were mostly studied in the limit of generalized dimensional reduction. Only recently the importance of keeping track of some crucial higher-dimensional features of these compactifications was fully appreciated. For example, in connection with the breaking of local symmetries, an important feature of the Scherk-Schwarz mechanism is the non-locality of the order parameter [2] (for a recent, pedagogical discussion see also [3]), which improves the ultraviolet behavior of symmetry-breaking quantities [4]. Such behavior would be out of theoretical control if one were to consider the reduced four-dimensional effective theory for the light modes only, instead of the compactified higher-dimensional one with its full tower of Kaluza-Klein excitations.

Recently, a generalized formulation of the Scherk-Schwarz mechanism was discovered [5], in the context of five-dimensional (5D) field-theory orbifolds, which can generate localized mass terms for the higher-dimensional bulk fields at the orbifold fixed points. Noticeable applications were found to 'brane-induced' supersymmetry breaking [6] and to gauge symmetry breaking [7]. The resulting mass spectrum is characterized by a universal shift of the Kaluza-Klein levels, exactly as in the conventional formulation. Also, when applied to minimal 5D Poincaré supergravity, both the conventional and the new formulation lead, at the classical level, to vanishing vacuum energy, undetermined gravitino masses (with a flat direction associated to the compactification radius $R$), and goldstinos along the internal components of the 5D gravitinos. Moreover, in both cases the mass shifts are controlled by a global parameter, which for brane-induced supersymmetry breaking can be identified with the overall jump of the gravitino field across the orbifold fixed points. All these analogies suggest that the two formulations may represent two equivalent descriptions of the same physical system, and that this equivalence may survive in the presence of interactions. This intepretation may sound counterintuitive. It is often assumed that the Scherk-Schwarz mechanism must correspond to a $y$-independent mass term in a basis of periodic 5D fields. It may also be tempting to associate the 'profiles' of $y$-dependent mass terms, directly linked to the shapes of the mass eigenmodes, to some intrinsic physical property of the system, and to expect that, when interactions are accounted for, different shapes would unavoidably lead to different transition rates and probabilities. We then believe that a systematic discussion of the structure of the interacting 5D effective theories, originated by Scherk-Schwarz compactifications on field-theory orbifolds, is in order. This is the purpose of the present paper. In full agreement with [5, 6, 7], we confirm the possibility of different but equivalent forms for the mass terms in otherwise identical effective theories, as a consequence of the freedom of performing local field redefinitions without changing the physics. We classify the conditions to be satisfied by the 5D mass terms, in a basis of periodic fields, in order to be fully ascribed to a Scherk-Schwarz twist, and we discuss how the equivalence continues to hold in the presence of interactions.

The plan of the paper is the following. For simplicity, and for an easier contact with the

the orbifold $S^1/Z_2$ with twisted periodicity conditions. After recalling the general formalism [1] and some well-known consistency conditions [8, 9, 10] for such a construction, we discuss the allowed structures for the coordinate-dependent mass terms in the corresponding effective 5D theories, formulated in a general basis of canonically normalized periodic fields. In particular, we show that the standard formalism, in which the Scherk-Schwarz twist is converted into a constant and $Z_2$-even 5D mass term, corresponds to the choice of a special basis of periodic fields within an infinite class. Conversely, we show that, given an effective 5D theory with periodic fields and an allowed set of coordinate-dependent mass terms, we can move to an equivalent 5D theory with no mass terms and a suitable Scherk-Schwarz twist. We conclude by mentioning how the equivalence can be generalized to a very large class of theories with arbitrary field content and interactions. As an example, we re-discuss the application to brane-induced supersymmetry breaking [6] in 5D Poincaré supergravity [11], and comment on its relation with gaugino condensation [12, 13, 14, 15] in M-theory [16]. Some useful definitions and formulae on path-ordered integrals are collected in the Appendix.

## 2. Twisted periodicity conditions for a 5D massless fermion on $S^1/Z_2$

We consider a 5D field theory compactified on the orbifold $S^1/Z_2$. We gauge-fix the invariance under 5D general coordinate transformations and choose the space-time coordinates $x^M \equiv (x^m, y)$ in such a way that the background metric is the Minkowski one, $\eta_{MN} = diag(-1,+1,+1,+1,+1)$. Similarly, we gauge-fix the local 5D Lorentz invariance to put the background value of the fünfbein in standard form, $e_M^{\ A} = \delta_M^A$. We can represent the orbifold on the whole real axis, identifying points related by a translation $T$ and a reflection $Z_2$ about the origin:

$$T: \ y \longrightarrow y + 2\pi R \, , \qquad\qquad Z_2: \ y \longrightarrow -y \, , \tag{1}$$

where $R$ is the radius of $S^1$.

For definiteness, we focus here on the theory of a 5D massless spinor $\Psi(x^m, y)$, and ignore the gravitational degrees of freedom associated with the fluctuations of the 5D metric. As will be discussed in the final section, our results can be easily extended to more general situations, including the case of the gravitino field in 5D supergravity, relevant for the discussion of local supersymmetry breaking. In terms of representations of the four-dimensional (4D) Poincaré group, the residual invariance of the chosen space-time background, the 5D spinor [1] $\Psi$ consists of two Weyl spinors $\psi_i$ $(i = 1, 2)$:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{2}$$

[1]From now on, the $x^m$-dependence of the fields will be always understood, and their $y$-dependence indicated only when appropriate. Notice that we differ from other frequently used notations in which $\Psi$ is represented by a Dirac spinor ($\psi_2 \rightarrow \overline{\psi_2}$) or by a symplectic Majorana spinor. We also define $\overline{\Psi} \equiv (\overline{\psi_1} \ \overline{\psi_2})^T$. Our 4D conventions are the same as in [17].

The system under consideration is described by the 5D Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \, . \tag{3}$$

In Eq. (3), $\mathcal{L}_0$ represents the free massless Lagrangian for $\Psi$,

$$\mathcal{L}_0 = i\overline{\Psi}^T \overline{\sigma}^m \partial_m \Psi - \frac{1}{2}\left( i\, \Psi^T \widehat{\sigma}^2 \partial_y \Psi + \text{h.c.} \right) , \tag{4}$$

where here and in the following we will denote with a hat the Pauli matrices acting on objects such as the one in Eq. (2). The remaining part of the 5D Lagrangian of Eq. (3), $\mathcal{L}_{int}$, contains possible interaction terms for the field $\Psi$, and in general may depend on additional 5D fields.

With respect to the $Z_2$ reflection that defines the orbifold, we will adopt the parity assignment

$$\Psi(-y) = Z\ \Psi(y) \, , \qquad Z = \widehat{\sigma}^3 \, . \tag{5}$$

For the consistency of the orbifold construction, both $\mathcal{L}_0$ and $\mathcal{L}_{int}$ must be invariant under $Z_2$, when suitable parities are assigned to the fields other than $\Psi$ appearing in $\mathcal{L}_{int}$. The free Lagrangian $\mathcal{L}_0$ is also invariant under

$$\Psi'(y) = U\, \Psi(y) \, , \tag{6}$$

where $U$ is a global $SU(2)$ transformation. We require that also $\mathcal{L}_{int}$ is $SU(2)$ invariant (after assigning suitable $SU(2)$ transformation properties to the fields other than $\Psi$) and does not contain derivatives of the field $\Psi$ or of other fields with non-trivial $SU(2)$ transformation properties.

In this framework, the field $\Psi(y)$ does not need to be periodic in $y$. It can be periodic up to a global SU(2) transformation, with 'twisted' periodicity conditions [1]:

$$\Psi(y + 2\pi R) = U_{\vec\beta} \Psi(y) \, , \qquad U_{\vec\beta} \equiv e^{i\vec\beta\cdot\vec\sigma} = \cos\beta\, \mathbf{1} + \frac{\sin\beta}{\beta}\, \vec\beta \cdot \vec\sigma \, , \tag{7}$$

where $\vec\sigma = (\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3)$, $\vec\beta = (\beta^1, \beta^2, \beta^3)$ is a triplet of real parameters, $\beta \equiv \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}$, and it is not restrictive to assume $\beta \leq \pi$. The operators $Z$ and $U_{\vec\beta}$ acting on the fields should provide representations of the space-time operations $T$ and $Z_2$ of Eq. (1). This gives rise to the well-known consistency condition [8, 9, 10]

$$U_{\vec\beta}\, Z\, U_{\vec\beta} = Z \, , \tag{8}$$

and for $Z = \widehat{\sigma}^3$ implies [2]

$$\vec\beta = (\beta_1, \beta_2, 0) \, . \tag{9}$$

The 4D modes have a spectrum characterized by a universal shift of the Kaluza-Klein levels, with respect to the mass eigenvalues $n/R$ of the periodic case, controlled by $\beta$:

$$m = \frac{n}{R} - \frac{\beta}{2\pi R} \, , \qquad (n \in \mathbf{Z}) \, . \tag{10}$$

[2]The choice $\vec\beta = (0, 0, \pi)$ gives $U_{\vec\beta} = -\mathbf{1}$, as any other choice with $\beta = \pi$.

conditions of Eq. (7):

$$\Psi^{(n)}(y) = \chi(x)\, e^{i\,\gamma\,\widehat{\sigma}^3} \begin{pmatrix} \cos my \\ \sin my \end{pmatrix}, \tag{11}$$

where $\chi(x)$ is $y$-independent 4D Weyl spinor satisfying the equation $i\sigma^m \partial_m \overline{\chi} = m\,\chi$, and, barring the trivial case $\vec{\beta} = 0$ in which the rotation angle $\gamma$ is arbitrary:

$$\gamma = \frac{1}{2} \arctan\left(\frac{\beta_1}{\beta_2}\right) + \delta + \rho\pi\,, \qquad \delta = \begin{cases} 0 & \text{for} \quad \beta_2 \geq 0 \\ \pi/2 & \text{for} \quad \beta_2 < 0 \end{cases}, \qquad (\rho \in \mathbf{Z})\,. \tag{12}$$

All this is well-known, it was reported here only for completeness and to make the following discussion more transparent.

## 3. Generation of 5D mass terms for periodic fields

We now move to a class of equivalent descriptions of the system characterized so far by the Lagrangian of Eqs. (3) and (4) and by the twisted periodicity conditions of Eqs. (7)-(9). Exploiting the fact that S-matrix elements do not change if we perform a local and non-singular field redefinition (see, e.g., [18]), we replace the twisted fields $\Psi(y)$ by periodic ones $\widetilde{\Psi}(y)$:

$$\Psi(y) = V(y)\,\widetilde{\Psi}(y)\,, \qquad \widetilde{\Psi}(y + 2\pi R) = \widetilde{\Psi}(y)\,, \tag{13}$$

where $V(y)$ must then be a $2 \times 2$ matrix satisfying

$$V(y + 2\pi R) = U_{\vec{\beta}}\, V(y)\,, \tag{14a}$$

as can be immediately checked from Eqs. (7) and (13). Besides condition (14a), we will impose for our convenience two additional constraints on the matrix $V(y)$. One is

$$V(y) \in SU(2)\,, \tag{14b}$$

which guarantees that the redefinition is non-singular, and that the kinetic terms for $\widetilde{\Psi}(y)$ remain canonical, as in $\mathcal{L}_0$. Moreover, the interaction terms in $\mathcal{L}_{int}$ are not modified, even if the $SU(2)$ transformation is $y$-dependent, as long as they do not involve derivatives of $\Psi$ or of other fields with non-trivial $SU(2)$ transformation properties. Thus, new terms can only originate from $\mathcal{L}_0$, when the $y$-derivative acts on $V(y)$. We also require that the new fields $\widetilde{\psi}_1(y)$ and $\widetilde{\psi}_2(y)$ have the same parities as the original ones $\psi_1(y)$ and $\psi_2(y)$:

$$\begin{cases} V_{ij}(-y) & = & +V_{ij}(y) & \qquad (ij = 11, 22) \\ V_{ij}(-y) & = & -V_{ij}(y) & \qquad (ij = 12, 21) \end{cases}. \tag{14c}$$

Notice that Eq. (14c) implies $V(0) = \exp\left(\,i\,\theta\,\widehat{\sigma}^3\,\right)$, with $\theta \in \mathbf{R}$.

Before exploring the effects of the field redefinition of Eq. (13), we observe that the solution to the conditions (14) is by no means unique. Starting from any given solution $V(y)$, a new set of solutions $V'(y)$ can be generated via matrix multiplication:

$$V'(y) = W_L(y)\,V(y)\,W_R(y)\,, \tag{15}$$

$$W_L(y+2\pi R)\, U_{\vec\beta} = U_{\vec\beta}\, W_L(y)\,, \qquad W_R(y+2\pi R) = W_R(y)\,, \tag{16a}$$

$$W_{L,R}(y) \in SU(2)\,, \tag{16b}$$

$$\left\{ \begin{array}{lclc} (W_{L,R})_{ij}(-y) & = & +(W_{L,R})_{ij}(y) & \qquad (ij=11,22) \\ (W_{L,R})_{ij}(-y) & = & -(W_{L,R})_{ij}(y) & \qquad (ij=12,21) \end{array} \right. . \tag{16c}$$

We are now ready to explore the effects of the field redefinition of Eq. (13). The Lagrangian $\mathcal{L}$, expressed in terms of the periodic field $\widetilde\Psi(y)$, describes exactly the same physics as before, but its form is now different:

$$\begin{aligned} \mathcal{L}(\Psi,\partial\Psi) &= \mathcal{L}(\widetilde\Psi,\partial\widetilde\Psi) + \left\{ -\frac{i}{2}\,[m_1(y)+i\,m_2(y)]\,\widetilde\psi_1\widetilde\psi_1 \right. \\ &+ \left. \frac{i}{2}\,[m_1(y)-i\,m_2(y)]\,\widetilde\psi_2\widetilde\psi_2 + i\,m_3(y)\,\widetilde\psi_1\widetilde\psi_2 + \text{h.c.} \right\}\,, \end{aligned} \tag{17}$$

where the mass terms $m_a(y)$ $(a=1,2,3)$ are the coefficients of the Maurer-Cartan form

$$m(y) \equiv m_a(y)\,\widehat\sigma^a = -i\,V^\dagger(y)\partial_y V(y)\,, \tag{18}$$

and satisfy:

$$m_a(y+2\pi R) = m_a(y)\,, \tag{19a}$$

$$m_a(y) \in \mathbf{R}\,, \tag{19b}$$

$$m_{1,2}(-y) = +m_{1,2}(y)\,, \qquad m_3(-y) = -m_3(y)\,, \tag{19c}$$

$$\begin{aligned} U_{\vec\beta} &= V(0)\,P\left[\exp\left(i\int_0^y dy' m(y')\right)\right]\,P_<\left[\exp\left(i\int_y^{y+2\pi R} dy' m(y')\right)\right]\times \\ P'&\left[\exp\left(-i\int_0^y dy' m(y')\right)\right]V^\dagger(0)\,, \qquad \left\{ \begin{array}{lll} P=P_< & P'=P_> & \text{for} \quad y>0 \\ P=P_> & P'=P_< & \text{for} \quad y<0 \end{array}\right. . \end{aligned} \tag{19d}$$

Properties (19a)-(19c) are in one-to-one correspondence with conditions (14a)-(14c) on $V(y)$. Eq. (19d) is related with Eq. (14a), and prescribes how the information on the twist of the original fields $\Psi(y)$ is encoded in the new Lagrangian. The symbols $P_<$ and $P_>$ denote inequivalent definitions of path-ordering, specified in the Appendix with some useful properties and the proof of Eq. (19d). Notice that, by taking the trace of both members in Eq. (19d), we obtain a relation between the Wilson loop and the twist parameter $\beta$:

$$\cos\beta = \frac{1}{2}\mathrm{tr}\,U_{\vec\beta} = \frac{1}{2}\mathrm{tr}\,P_<\left[\exp\left(i\int_y^{y+2\pi R} dy' m(y')\right)\right]\,. \tag{20}$$

Notice also that, because of the freedom of performing global $SU(2)$ transformations with the constant matrix $V(0)$, which are invariances of the Lagrangian, different values of the twist $\vec\beta$ with the same value of $\beta$ correspond to physically equivalent descriptions.

The mass terms in Eq. (17) are of three different types, associated with the three possible bilinears [3] $\widetilde{\psi}_1\widetilde{\psi}_1$, $\widetilde{\psi}_2\widetilde{\psi}_2$ and $\widetilde{\psi}_1\widetilde{\psi}_2$. Because of Eq. (19), they do not correspond to the most general set of $y$-dependent mass terms allowed by 4D Lorentz invariance, which would be characterized by three independent complex functions. The rôle of the conditions (19) is to guarantee the equivalence between the descriptions on the two sides of Eq. (17). We stress again that this equivalence is not limited to the free-field case, but also holds true in the interacting case [4].

Also the converse is true. Given a Lagrangian such as the one on the right-hand side of Eq. (17), expressed in terms of periodic fields $\widetilde{\psi}_i(y)$ $(i = 1, 2)$ and with mass terms satisfying Eq. (19), we can move to the equivalent Lagrangian of Eqs. (3) and (4), where all mass terms have been removed, and the fields satisfy the generalized periodicity conditions of Eq. (7), by performing the field redefinition of Eq. (13). As shown in the Appendix, $V(y)$ is given by:

$$V(y) = V(0)\, P\left[\exp\left(i \int_0^y dy' m(y')\right)\right] , \qquad P = \begin{cases} P_< & \text{for} \quad y > 0 \\ P_> & \text{for} \quad y < 0 \end{cases} . \tag{21}$$

For any $V(0) = \exp\left(i\,\theta\,\widehat{\sigma}^3\right)$ $(\theta \in \mathbf{R})$, conditions (14) are satisfied with $U_{\vec{\beta}}$ given by Eq. (19d). The arbitrariness in $V(0)$ reflects the fact that physically distinct theories are characterized by $\beta$, not by $\vec{\beta}$.

## 4. Examples and localization of 5D mass terms

From the discussion of the previous section, it is clear that mass 'profiles' $m_a(y)$ for periodic fields, of the type specified in Eq. (19), do not have an absolute physical meaning. They can be eliminated from the Lagrangian and replaced by a twist, the two descriptions being completely equivalent. Moreover, all Lagrangians with the same $\mathcal{L}_{int}$ and mass profiles corresponding to the same twist $\vec{\beta}$, as computed from Eq. (19d), are just different equivalent descriptions of the same physics. Indeed, suppose that $\mathcal{L}^1$ and $\mathcal{L}^2$ are two such Lagrangians, and call $V^I(y)$ $(I = 1, 2)$ the local redefinitions mapping $\mathcal{L}^I$ into the Lagrangian $\mathcal{L}$ for the twisted, massless 5D fields $\Psi(y)$. Then $\mathcal{L}^1$ and $\mathcal{L}^2$ are related by the local non-singular field redefinition $V(y) = V^{2\dagger}(y)V^1(y)$. This shows that, in the class of interacting models under consideration, what matters is the twist $\vec{\beta}$ and not the specific form of the mass terms $m_a(y)$ enjoying the properties (19) [5]. We can make use of this

[3] At variance with ref. [15], we find that the coefficients of the bilinears $\widetilde{\psi}_1\widetilde{\psi}_1$ and $\widetilde{\psi}_2\widetilde{\psi}_2$, although related, should not be necessarily equal. We also disagree with the statement in [15] that 'the Scherk-Schwarz mechanism is equivalent to the mechanism of adding a mass term only if this mass term is $Z_2$-even and constant'.

[4] Actually, the equivalence between two Lagrangians related by a local field redefinition holds irrespectively of the explicit form assumed by the interaction terms. From this point of view, we could drop the assumption that $\mathcal{L}$ does not contain derivatives acting on fields with non-trivial $SU(2)$ transformation properties such as $\Psi$. In this case, interaction terms involving $\partial_y\Psi$ would generate, via the redefinition of Eq. (13), additional but controllable contributions to the right hand side of Eq. (17).

[5] Actually, in view of the observations after Eqs. (20) and (21), what really matters is $\beta$.

without affecting the physical properties of the theory.

As an example, we consider the simple case in which the twist parameter is just

$$\vec{\beta} = (0, \beta, 0) \,. \tag{22}$$

Then a frequently used solution to Eq. (14), for the twist specified by Eq. (22), is

$$V^O(y) = \exp\left( i\, \beta \widehat{\sigma}^2 \frac{y}{2\pi R} \right) \,, \tag{23}$$

the symbol '$O$' standing for *'ordinary'*. Starting from the Lagrangian $\mathcal{L}^0$ for the periodic fields $\widetilde{\Psi}(y)$, defined by $V^O(y)$ via the redefinition of Eq. (13), and performing the standard Fourier decomposition of the 5D fields into 4D modes, we can immediately check that the 4D mass eigenvalues and eigenfunctions are indeed given by Eqs.(10)-(12), with $\gamma = \delta = 0$. Applying Eq. (18) to $V^O(y)$, we find the constant mass profile:

$$m_1^O(y) = m_3^O(y) = 0 \,, \qquad m_2^O(y) = \frac{\beta}{2\pi R} \,, \tag{24}$$

and we can check that, in agreement with Eq. (20):

$$\beta = \int_y^{y+2\pi R} dy' m_2^O(y') \,. \tag{25}$$

We now move, following [5], to a more general solution of Eqs. (14) and (22), where, in a basis of periodic fields, the system is described by a different Lagrangian $\mathcal{L}^G$ (the symbol '$G$' stands for *'generalized'*). $\mathcal{L}^G$ is still of the general form of Eq. (17), including interaction terms, but now:

$$m_1(y)^G = m_3^G(y) = 0 \,, \qquad m_2^G(y) \neq 0 \,, \tag{26}$$

and $m_2^G(y)$ is an otherwise arbitrary real, periodic, even function of $y$, with the property that

$$\int_y^{y+2\pi R} dy' m_2^G(y') = \int_y^{y+2\pi R} dy' m_2^O(y') = \beta \,. \tag{27}$$

As long as the above properties are satisfied, the two Lagrangians $\mathcal{L}^O$ and $\mathcal{L}^G$ are physically equivalent. Two representative and equivalent choices of $m_2^G(y)$ are illustrated in Fig. 1: the dashed line shows a mild (Gaussian) localization around the orbifold fixed points, the solid line a strong localization. The interactions between $\widetilde{\Psi}(y)$ and other fields are not determined by the shapes of the fermion eigenmodes and, indirectly, by the profile of $m_2(y)$. Neither the mass spectrum, nor the interactions depend on shapes, which are an artifact of the choice of field variables. As long as the twist is kept fixed, shapes can be arbitrarily deformed along $y$, without changing the physics.

A possible special choice for $m_2^G(y)$ is the singular limit:

$$m_2^G(y) = \sum_{q=-\infty}^{+\infty} \left[ \delta_0 \, \delta(y - 2q\pi R) + \delta_\pi \, \delta(y - (2q+1)\pi R) \right] \,, \qquad \delta_0 + \delta_\pi = \beta \,, \tag{28}$$

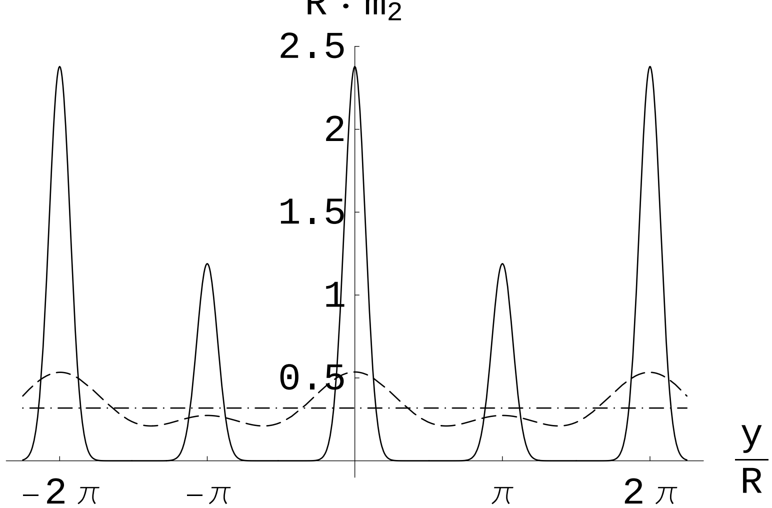


Figure 1: Two representative and equivalent choices for $m_2^G(y)$, corresponding to $\beta = 2$. For reference, the dash-dotted line shows the equivalent constant profile $m_2^O(y) = 1/(\pi R)$.

where what we actually mean is, as discussed in detail in [5, 6], a suitably regularized version of the distribution in Eq. (28). This description is apparently quite remote from the 'ordinary' one. The mass terms vanish everywhere but at the orbifold fixed points, where there are localized contributions to $m_2(y)$. The redefinitions bringing from the massive periodic fields of $\mathcal{L}^O$ and $\mathcal{L}^G$ to the corresponding massless twisted 5D fields are:

$$\Psi(y) = V^{O,G}(y)\ \widetilde{\Psi}^{O,G}(y)\,, \tag{29}$$

with $V^O(y)$ given by Eq. (23) and

$$V^G(y) = \exp\left[i\,\alpha(y)\widehat{\sigma}^2\right]\,. \tag{30}$$

Here $\alpha(y)$, depicted in Fig. 2 for some representative choices of $\delta_0$ and $\delta_\pi$, is given by:

$$\alpha(y) = \frac{\delta_0 - \delta_\pi}{4}\ \epsilon(y) + \frac{\delta_0 + \delta_\pi}{4}\ \eta(y)\,, \tag{31}$$

where the function $\epsilon(y)$ is the periodic sign function and $\eta(y)$ is the 'staircase' function

$$\eta(y) = 2q + 1\,, \qquad q\pi R < y < (q+1)\pi R\,, \qquad (q \in \mathbf{Z})\,. \tag{32}$$

The local field redefinition that relates the two Lagrangians $\mathcal{L}^O$ and $\mathcal{L}^G$ is [6]:

$$\widetilde{\Psi}^G(y) = V^{G\dagger}(y)V^O(y)\ \widetilde{\Psi}^O(y)\,. \tag{33}$$

Notice that the periodic fields $\widetilde{\Psi}^G(y)$ are not smooth but only piecewise smooth [5]. This can be checked either by integrating the equations of motion for $\widetilde{\Psi}^G(y)$, derived from $\mathcal{L}^G$,

[6] On the basis of the equivalence between $\mathcal{L}^O$ and $\mathcal{L}^G$ shown here, we disagree with the statement of ref. [15] that 'the Scherk-Schwarz mechanism is equivalent to the mechanism of adding a mass term . . . for sure not when the mass terms are localized at the fixed points of the orbifold'.

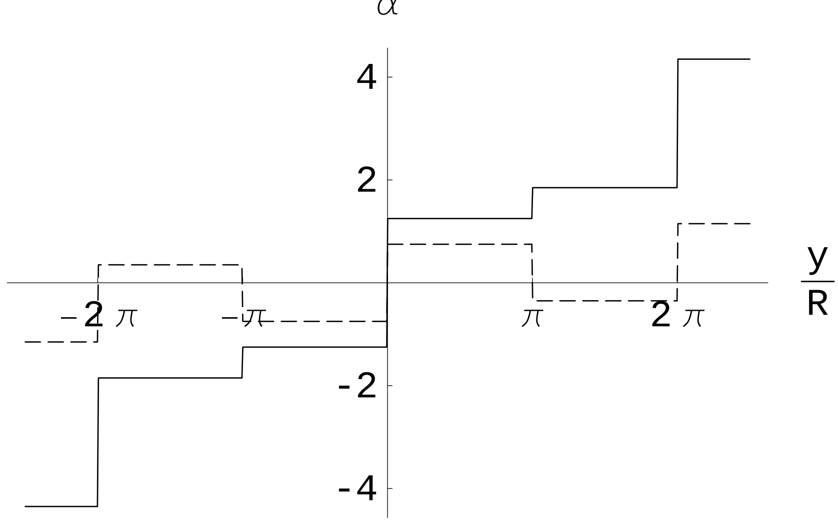


Figure 2: The function $\alpha(y)$ for two representative parameter choices: the solid line corresponds to $\delta_0 = 2.5$, $\delta_\pi = 0.6$, the dashed one to $\delta_0 = 1.5$, $\delta_\pi = -1.1$.

in a small region around the fixed points [5], or by making use of the field redefinition in Eq. (33), recalling that $\widetilde{\Psi}^O(y)$ and $V^O(y)$ are smooth while $V^G(y)$ is not. We find that the fields $\widetilde{\Psi}^G(y)$ have cusps and discontinuities described by:

$$\left\{ \begin{array}{l} \widetilde{\Psi}^G(2q\pi R + \xi) = e^{i\,\delta_0 \sigma^2}\, \widetilde{\Psi}^G(2q\pi R - \xi) \\ \widetilde{\Psi}^G[(2q+1)\pi R + \xi] = e^{i\,\delta_\pi \sigma^2}\, \widetilde{\Psi}^G[(2q+1)\pi R - \xi] \end{array} \right. , \qquad (0 < \xi \ll 1\,,\, q \in \mathbf{Z})\,, \tag{34}$$

where the 'jumps' of the field variables are parametrized by $\delta_{0,\pi}$.

Another simple but instructive example corresponds to a Lagrangian for periodic fields of the form in Eq. (17), where now

$$m_1(y) = m_2(y) = 0\,, \qquad m_3(y) \neq 0\,, \tag{35}$$

and $m_3(y)$ is an otherwise arbitrary real, odd, periodic function of $y$, as prescribed by Eqs. (19a)-(19c). Notice that, for any such function, Eq. (20) gives always $\beta = 0$, since

$$\int_y^{y+2\pi R} dy' m_3(y') = 0\,. \tag{36}$$

In other words, real, periodic, odd mass profiles can be completely removed by a field redefinition without introducing a non-trivial twist. Such a field redefinition corresponds to:

$$V(y) = \exp\left[ i \int_0^y dy' m_3(y') \right]\,. \tag{37}$$

Some representative profiles for $m_3(y)$ are exhibited in Fig. 3. Notice that no constant $m_3(y) \neq 0$ is allowed by Eqs. (19a)-(19c), and also a $m_3(y) \neq 0$ completely localized at $y = 2q\pi R$ and/or $y = (2q+1)\pi R$ is forbidden. An allowed possibility is a piecewise constant $m_3(y)$, for example:

$$m_3(y) = \mu\, \epsilon(y)\,, \tag{38}$$

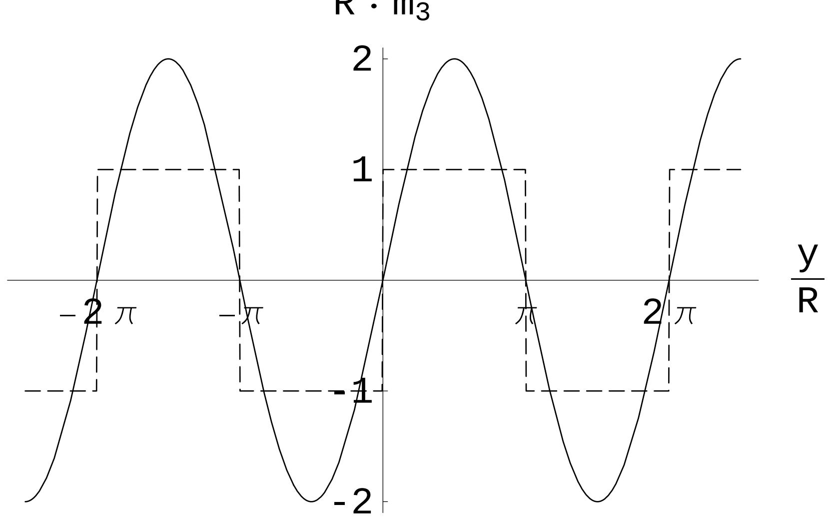


Figure 3: Two representative and equivalent choices for $m_3(y)$: the solid line corresponds to $m_3(y) = (2 \sin y)/R$, the dashed one to $m_3(y) = \epsilon(y)/R$.

where $\epsilon(y)$ is the periodic sign function and $\mu$ a real constant with the dimension of a mass.

Before concluding this section, it is appropriate to comment on the relation between our $m_3(y)$ and other odd mass terms discussed in related but different frameworks. Ref. [15] discussed odd mass terms that do not satisfy, in our notation, Eq. (19b). Therefore, the fact that those mass terms give a spectrum different from the one of Eq. (10) is not in contradiction with our results: we have already stressed in Section 3 that the mass terms compatible with the Scherk-Schwarz mechanism do not correspond to the most general set of $y$-dependent mass terms allowed by 4D Lorentz invariance.

## 5. Generalizations and discussion

The results of the previous sections can be suitably generalized to 5D theories with a general content of bosons and fermions, general interactions, and general symmetries to be exploited for the Scherk-Schwarz twist, as already discussed on some examples [6, 7]. A particularly interesting case would be the effective 5D supergravity corresponding to M-theory [16] compactified on a small Calabi-Yau manifold times a large orbifold $S^1/Z_2$. In particular, it would be interesting to explore further the intriguing analogies [12, 10] between non-perturbative supersymmetry breaking via gaugino condensation at the orbifold fixed points [12, 13, 14] and supersymmetry breaking via the Scherk-Schwarz mechanism, in view of our generalized description [5, 6] of the latter. The effective 5D supergravity of M-theory is a quite complicated one, because of the presence of a warped background and of different types of multiplets, including some compactification moduli. Here we discuss only a toy example, namely pure 5D Poincaré supergravity, showing that the general effective theory of Scherk-Schwarz supersymmetry breaking can indeed encompass [6] the analogies and the differences with the full-fledged M-theory.

In the on-shell Lagrangian of pure $N = 1$, $D = 5$ Poincaré supergravity [11], the 5D spinor of Eq. (2) is replaced by the 5D gravitino, in our notation:

$$\Psi_M = \begin{pmatrix} \psi_{1M} \\ \psi_{2M} \end{pmatrix}, \qquad (M = m, 5) . \tag{39}$$

The bosonic superpartners of the gravitino are the fünfbein $E_M^A$ and the graviphoton $B_M$, singlets under the global $SU(2)$ (R-)symmetry under which $\Psi_M$ transforms as in Eq. (6), with the same constant matrix $U$ for every value of the index $M$. The $Z_2$ parity assignments to the fermionic fields are now $Z = \hat{\sigma}^3$ for $\Psi_m$, $Z = -\hat{\sigma}^3$ for $\Psi_5$, and can be consistently completed by assigning even parity to $(E_m^a, E_5^5, B_5)$ and odd parity to $(E_m^5, E_5^a, B_m)$. We expand the theory around a flat background, solution of the 5D equations of motion:

$$\langle E_M^A \rangle = \delta_M^A , \qquad \langle \Psi_M \rangle = \langle B_M \rangle = 0 . \tag{40}$$

The implementation of the Scherk-Schwarz twist on $S^1/Z_2$ and the derivation of the corresponding effective 5D theory for periodic fields can be discussed in parallel with Sections 2 and 3, thus we do not repeat all the details. Here we just comment on the new features and on the resulting structure of 5D gravitino mass terms (complementary details can be found in [6]). The twisted boundary conditions on the gravitino can be written as:

$$\Psi_M(y + 2\pi R) = U_{\vec{\beta}} \, \Psi_M(y) , \qquad (M = m, 5) , \tag{41}$$

with $U_{\vec{\beta}}$ as in Eq. (7), and the consistency conditions of Eqs. (8) and (9) still apply. Similarly, the field redefinition bringing to the basis of periodic fields reads:

$$\Psi_M(y) = V(y) \, \widetilde{\Psi}_M(y) , \qquad (M = m, 5) , \tag{42}$$

where $V(y)$ satisfies as before the conditions of Eq. (14).

The only term of the Lagrangian involving derivatives of the gravitino is its generalized kinetic term, which also involves the degrees of freedom associated with the fünfbein. Therefore, when moving to the basis of periodic fields, not only mass terms but also some interaction terms will be generated. Since we are interested here in the structure of the gravitino mass terms, we set all the bosonic fields to their background values of Eq. (40) and focus on the fermion bilinears. Up to a universal normalization factor, the Lagrangian is

$$\mathcal{L} = -\frac{1}{2} \epsilon^{mnpq} \overline{\Psi}_m^T \overline{\sigma}_n \partial_p \Psi_q + \left( -\frac{i}{2} \Psi_m^T \sigma^{mn} \hat{\sigma}^2 \partial_y \Psi_n + i \Psi_5^T \sigma^{mn} \hat{\sigma}^2 \partial_m \Psi_n + \text{h.c.} \right) + \ldots . \tag{43}$$

After moving to the basis of periodic fields via the field redefinition of Eq. (42), we get:

$$\begin{aligned} \mathcal{L}(\Psi_M, \partial \Psi_M) &= \mathcal{L}(\widetilde{\Psi}_M, \partial \widetilde{\Psi}_M) + \left\{ -\frac{i}{2} \left[ m_1(y) + i \, m_2(y) \right] \widetilde{\psi}_{m1} \sigma^{mn} \widetilde{\psi}_{n1} \right. \\ &+ \left. \frac{i}{2} \left[ m_1(y) - i \, m_2(y) \right] \widetilde{\psi}_{2m} \sigma^{mn} \widetilde{\psi}_{2n} + i \, m_3(y) \, \widetilde{\psi}_{1m} \sigma_{mn} \widetilde{\psi}_{2n} + \text{h.c.} \right\} . \end{aligned} \tag{44}$$

From here on the discussion can proceed as in the simpler case of the previous sections. The only important difference is that gravitino masses occur via the super-Higgs effect, with the goldstino components provided by $\widetilde{\Psi}_5$. To discuss the spectrum in the case of a non-trivial Scherk-Schwarz twist, $\beta \neq 0$, it is convenient to go to the unitary gauge, where $\widetilde{\Psi}_5$ completely disappears from the Lagrangian on the right-hand side of Eq. (44). We could now repeat the whole discussion of Section 4. In particular, the case of Eqs. (26) and (28) corresponds to gravitino mass terms entirely localized at the orbifold fixed points,

$$\mathcal{L}_{mass}(\widetilde{\Psi}_M, \partial\widetilde{\Psi}_M) = \frac{1}{2}\left[\delta_0\, \delta(y) + \delta_\pi \delta(y - \pi R)\right] \left(\widetilde{\psi}_{m1}\sigma^{mn}\widetilde{\psi}_{n1} + \widetilde{\psi}_{m2}\sigma^{mn}\widetilde{\psi}_{n2} + \text{h.c.}\right) , \quad (45)$$

where we can interpret the constants $\delta_0$ and $\delta_\pi$ as the remnants of some localized brane dynamics, which may include gaugino condensation. We remind the reader that [5, 6] deriving the equations of motion and the mass spectrum in the presence of the localized Lagrangian of Eq. (45) requires a regularization [7].

We conclude with some comments on the possible extension of the previous considerations to the case of gaugino condensation in M-theory [12, 13, 14]. We recall that in such case localized gravitino mass terms can be induced into the effective 5D supergravity Lagrangian by the non-zero VEV of $G_{ABCD}$, the four-form of eleven-dimensional supergravity. The VEV related with the gaugino condensate, because of a perfect square structure that appears in the Lagrangian, is $\langle G_{11abc}\rangle$, where $a, b, c = 1, 2, 3$ are holomorphic indices associated with the six-dimensional Calabi-Yau manifold. $\langle G_{11abc}\rangle$ is even under the $Z_2$ parity, and generates gravitino mass terms of the type of $m_2(y)$ in Eq. (44). The other possible VEV, $\langle G_{a\bar{a}b\bar{b}}\rangle$, is odd under the $Z_2$ parity, and generates gravitino mass terms of the form of $m_3(y)$ in Eq. (44). However, now $m_3(y)$ is imaginary rather than real as dictated by Eq. (19b). As discussed in [15], such a term may give a spectrum different from the Scherk-Schwarz one. Naively, this would suggest that, in the presence of a non-vanishing $\langle G_{a\bar{a}b\bar{b}}\rangle$, gaugino condensation in M-theory cannot be reinterpreted as a generalized Scherk-Schwarz mechanism. However, there are other effects that play a role. For example, the odd mass terms generated by $\langle G_{a\bar{a}b\bar{b}}\rangle$ only appear in the intermediate steps of the derivation of the effective 5D theory. In the final form of the resulting 5D supergravity Lagrangian, as written for example in [14], the contribution of those mass terms cancels, after an integration by parts, against a contribution originated by the non trivial $y$-dependence of some moduli fields. We conclude that the possible equivalence of gaugino condensation in M-theory with a generalized Scherk-Schwarz mechanism is still an open issue, whose complete clarification requires further work.

[7] Alternatively, one could use an equivalent localized Lagrangian where only bilinears in the even fields do appear, and to which one can apply the naive variational principle without regularization. The precise meaning of such a Lagrangian is discussed in [5, 6], here we just stress that it cannot be obtained from (45) by means of an $SU(2)$ transformation.

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We collect here some useful formulae and results concerning path-ordered products, and show how they can be used to prove Eq. (21) and Eq. (19d). First, we distinguish the two inequivalent definitions of path-ordering, introducing the symbols:

$$\begin{aligned} P_>[m(y_1)m(y_2)] &\equiv m(y_1)m(y_2)\Theta(y_1-y_2)+m(y_2)m(y_1)\Theta(y_2-y_1)\,, \\ P_<[m(y_1)m(y_2)] &\equiv m(y_1)m(y_2)\Theta(y_2-y_1)+m(y_2)m(y_1)\Theta(y_1-y_2)\,, \end{aligned} \tag{46}$$

where $m(y)$ is a $y$-dependent matrix and

$$\Theta(y) = \begin{cases} 1 & \text{for} \quad y>0 \\ 0 & \text{for} \quad y<0 \end{cases} \tag{47}$$

is the Heaviside step function. From the above definitions, and assuming $y_1 < y_2 < y_3$, the following properties follow:

$$P_>\left[\exp\left(i\int_{y_1}^{y_2} dy' m(y')\right)\right] \cdot P_<\left[\exp\left(-i\int_{y_1}^{y_2} dy' m(y')\right)\right] = \mathbf{1}\,, \tag{48}$$

$$P_>\left[\exp\left(i\int_{y_1}^{y_3} dy' m(y')\right)\right] = P_>\left[\exp\left(i\int_{y_2}^{y_3} dy' m(y')\right)\right] \cdot P_>\left[\exp\left(i\int_{y_1}^{y_2} dy' m(y')\right)\right]\,, \tag{49}$$

$$P_<\left[\exp\left(i\int_{y_1}^{y_3} dy' m(y')\right)\right] = P_<\left[\exp\left(i\int_{y_1}^{y_2} dy' m(y')\right)\right] \cdot P_<\left[\exp\left(i\int_{y_2}^{y_3} dy' m(y')\right)\right]\,. \tag{50}$$

If the $y$-dependent matrix $V(y)$ satisfies the differential equation:

$$\partial_y V(y) = i\, V(y)\, m(y)\,, \tag{51}$$

then it is immediate to prove that

$$V(y) = V(y_0)\, P\left[\exp\left(i\int_{y_0}^{y} dy' m(y')\right)\right]\,, \qquad P = \begin{cases} P_< & \text{for} \quad y_0 < y \\ P_> & \text{for} \quad y_0 > y \end{cases}\,. \tag{52}$$

Correspondingly, if $m(y)$ is hermitian, then $V^\dagger(y)$ obeys the equation

$$\partial_y V^\dagger(y) = -i\, m(y)\, V^\dagger(y)\,, \tag{53}$$

which is solved by

$$V^\dagger(y) = P\left[\exp\left(-i\int_{y_0}^{y} dy' m(y')\right)\right] V^\dagger(y_0)\,, \qquad P = \begin{cases} P_> & \text{for} \quad y_0 < y \\ P_< & \text{for} \quad y_0 > y \end{cases}\,. \tag{54}$$

Showing that Eq. (18) implies Eq. (21) is now a simple application of Eqs. (51) and (52). To show instead that Eqs. (14a) and (18) imply Eq. (19d), it is sufficient to solve Eq. (14a) for $U_{\vec\beta}$,

$$U_{\vec\beta} = V(y+2\pi R)V^\dagger(y)\,, \tag{55}$$

and to insert the explicit form of the solutions of Eq. (18), namely Eqs. (52) and (54). It is also easy to show that the second member of Eq. (19d) is indeed $y$-independent.

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